STORAGE ANALYSIS OF SEASONAL NET INPUTS*

by

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1. Introduction

The inherent fluctuations of hydrologic phenomena such as the natural discharge of rivers have shown significant impacts on the social, economic and political activities of man. Their effects become even more intensified due to the rapidly increasing population all competing for the limited water resources available. Aside from the natural variability of hydrologic variables, the temporal and spatial distribution of water resources determines their full or limited utilization to satisfy the demand.

Water storage is the basic means for providing water transfers in time and space through which the supplies and the demands can be matched. Numerous water storage structures have been constructed to make optimal and efficient use of available water resources. As the number and the size of these structures increase, a need arises to develop a more refine and efficient design procedures. Thus, these procedures have evolved from simple empirical techniques to more elaborate and accurate techniques. A branch of hydrology that evolved looking for solutions to water storage problems is the stochastic theory of storage (Lloyd, 1967; and Klemes, 1981).

One problem of particular interest in storage theory involves the analysis of non-stationary periodic stochastic processess of inflows and outflows in determining the required storage capacity of

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a reservoir. Several methods have been developed such as the so-called sequent-peak algorithm (Thomas and Fiering, 1963; and Fiering, 1965). The analytical treatment of water storage capacities of periodic-stochastic inflows and outflows using range analysis of cumulative (partial) sums of random variables has also been dealt with elsewhere (Salas, 1972).

One analytical approach in determining the storage capacity of a reservoir is based on the concept of the maximum accumulated deficit of partial sums (Yevjevich, 1965). Deficit analysis of stationary random variables has been shown by Gomide (1975) to follow directly from the theory of Markov chains.

This paper deals with deficit analysis of seasonal series. The preceding section presents a quick review of previous works related to the study. A concise but thorough development of the numerical framework of the algorithm used in deficit analysis is given in section 3. The analytical and simulation results are given in section 4. The paper ends with a summary and conclusions.

2. *Theoretical Background*

The basic theory and properties of Markov chains can be found in statistical books such as those by Feller (1968) and Kemeny and Snell (1974). A comprehensive presentation on the subject is given by Lloyd (1974). This section describes the basic structure and properties of Markov chains, and then presents their applications to solving practical storage problems.

2.1 *Markov Chains and Finite Reservoir Theory*

Consider a univariate discrete process x_t representing a random variable at time epoch t. A model in which the distribution of x_t at time epoch t is influenced by the k most recent realizations only is called a k-step Markov chain; that is,

$$P(x_t=i \mid x_{t-1}=j,\ x_{t-2}=\ell, \ldots, x_1=u, x_0=v)$$
$$= P(x_t=i \mid x_{t-1}=j, x_{t-2}=\ell, \ldots, x_{t-k}=m) \qquad (1)$$

For the particular case of simple Markov chain, the variable x_t is influenced only by the previous realization of x_{t-1}, and nothing else. Thus, eq. (1) reduces to

$$P\,(x_t=i \quad x_{t-1}=j, x_{t-2}=\ell, \ldots, x_1=u, x_0=v)$$
$$= P\,(x_t=i \quad x_{t-1}=j) \tag{2}$$

The stochastic reservoir theory borrows much from the theory of Markov chains for its practical applicability. Basically, it involves representing the flows in discrete units and time intervals and discretizing the reservoir volume into a number of parts, thus creating a system of equations that approximate the integral equations (Moran, 1954; 1959). The input to the reservoir is assumed to be a stationary sequence of mutually independent inflows, and the release rule governed by the current storage level. It can be shown that the sequence of storage level $[s_t]$ where s_t is the storage level at time epoch t, has a correlation structure of a Markov chain.

In the particular case of a reservoir with capacity K units and water release of M units, and assuming the so-called "simultaneous model" of inflows and outlows (Lloyd, 1963), then the matrix Q of transition probabilities of $[s_t]$ is given by

$Q=$

$t \backslash t-1$	0	1	2	3	...	k-1	k
0	F_M	F_{M-1}	F_{M-2}	F_{M-3}	.	o	o
1	f_{M+1}	f_M	f_{M-1}	$M-2$	.	o	o
2	f_{M+2}	f_{M+1}	M	.	⋮	⋮	
3	f_{M+3}	f_{M+2}	f_{M+1}				
o	o	o	o	o		o	o
k–1	f_{M+k-1}	f_{M+k-2}	f_{M+k-3}	f_{M+k-4}		f_M	f_{M-1}
k	G_{M+k}	G_{M+k-1}	G_{M+k-2}	G_{M+k-3}.		G_{M+1}	G_M

(3)

where $f_r=P(x_t=r)$, $F_r=P(X_t \leqslant r)$ and $G_r=p(X_t \geqslant r)$.

Relaxing the condition that the inflow distribution is stationary, let the inflows be seasonal with a different distribution for each season in the year. Assume for the moment that the inflows are mutually independent from one season to another. Thus, the matrix of eq. (3) can readily be extended to the case where the transition probabilities display seasonality, i.e. a matrix Q_T is defined for each season τ. The periodic matrices of transition probabilities account for the seasonal fluctuations of storage level of a reservoir.

The same methodology can be extended to seasonal correlated inputs whereby the associated transition matrices are constructed such that the season-to-season correlations of inflows are properly considered. In the case of correlated stationary (annual) inflows, Lloyd (1963) proposed the use of a simple bivariate Markov chain to represent the simultaneous distribution of inflow and storage level. On the other hand, Gomide (1975) represented the state of the system for correlated inputs as a two-step Markov chain. It may be mentioned that the former approach is more general and convenient than the latter since the latter approach presents difficulties when the input assumes more than two values. In this study, the procedure by Lloyd (1963) will be adopted to treat the case of seasonal correlated inflows.

2.2 Deficit Analysis Using Markov Chains

One approach to determine the required storage capacity of a reservoir is by the maximum accumulated deficit analysis, or simply deficit analysis, which assumes a bottomless reservoir. Before describing the analytical approach used in this study, the following definitions are in order.

Let $[X_t]$ be a sequence of random variables, and $s_i = \sum_{t=1}^{i} x_t$, where i=1, 2, . . . n. The random variable s_i is called the cumulative or partial sum. The maximum partial sum M_n, the minimum partial sum m_n, and the maximum accumulated deficit of partial sums D_n for the sample size n are defined as

$$M_n = \max\,[0, s_1, s_2, \ldots, s_n]$$

$$m_n = \min\,[0, s_1, s_2, \ldots, s_n]$$

$$D_n = \max\,[0, M_1 - s_1\ M_2 - s_2, \ldots, M_n - s_n] \qquad (4)$$

In deficit analysis, the sequence $[s_i]$ is interpreted to represent the storage level in a semi-infinite reservoir which has a top but no bottom.

The analytical approach to deal with deficit analysis of stationary random variables has been shown by Gomide (1975) to follow directly from the theory of Markov chains. The distribution of the maximum deficit D_n can be determined using Markov chains when the state space is such that one boundary (empty reservoir) is absorbing and the other (full reservoir) is reflecting. In the particular case of independent stationary discrete net inputs, Gomide (1975) gave the distribution of D_n as

$$P\,[D_n \leqslant k] = \sum_{s=1}^{k+1} p_{k+1}^{(n)}\ (s, k+1) \qquad (5)$$

where the summation in the right-hand side of eq. (5) denotes "the probability that the system does not reach state zero in the first n-steps, given the initial state $u=k+1$". For correlated inputs, a quite different expression for $P\,(D_n \leqslant k)$ was obtained mainly because he represented the state of the system for dependent inputs as a two-step rather than as a simple bivariate Markov chain.

In this study it will be shown that the theory of deficit analysis can be extended for seasonal inputs whereby a different transition probability matrix is considered for each season. But due to computational constraints involved in obtaining the exact distributions of the maximum deficit for large sample size or finer discretization of input distributions, the analysis will be done for small sample sizes and for simple cases of seasonal series only. However, the basic approach applies equally well to a variety of cases.

3. Analytical Approach

This section describes the analytical approach to deficit analysis of periodic stochastic process of independent and correlated seasonal input series.

3.1 Independent Seasonal Inputs

Consider the case of an independent seasonal discrete random variable $[x_t]$ with $t=(\nu-1)w+\tau$, where τ denotes the season, ν denotes the year, and w the number of seasons in the year, such that $p\ (x_t=i) = p_t(i)$ where i is an element of the set $[\ -a, -a+1, \ldots, 0, \ldots, a\text{-}1, a]$ representing the discrete values of net inflow. It is assumed for convenience in treating matrices of the same size that the number of states are the same for each season but the transition probabilities may be different. Let $[x_t]$ be the net input at discrete time t into a reservoir of size k+1. It can be shown that the sequence of the amount of water stored $[s_t]$ follows as Markov chain with state space $[0, 1, 2, \ldots, k+1]$. Furthermore, let this reservoir be such that the empty state is an absorbing boundary and the full state is a reflecting boundary. Then, $[s_t]$ has periodic (seasonal) one-step transition matrix defined as:

	0	1	2	3	...	k	k+1
0	1	$\ell_\tau(-1)$	$\ell_\tau(-2)$	$\ell_\tau(-3)$	...	$\ell_\tau(-k)$	$\ell_\tau(-k-1)$
1	o	$p_\tau(0)$	$p_\tau(-1)$	$p_\tau(-2)$	...	$p_\tau(-k+1)$	$p_\tau(-k)$
2	o	$p_\tau(1)$	$p_\tau(0)$	$p_\tau(-1)$	...	$p_\tau(-k+2)$	$p_\tau(-k+1)$
$p^*_\tau\ (k+1) =$ 3	o	$p_\tau(2)$	$p_\tau(1)$	$p_\tau(0)$	...	$p_\tau(-k+3)$	$p_\tau(-k+2)$
⋮	⋮	⋮	⋮	⋮	·	⋮	⋮
k	o	$p_\tau(k-1)$	$p_\tau(k-2)$	$p_\tau(k-3)$	...	$p_\tau(0)$	$p_\tau(-1)$
k+1	o	$u_\tau(k)$	$u_\tau(k-1)$	$u_\tau(k-2)$	...	$u_\tau(1)$	$u_\tau(0)$

where

$$u_\tau(j) = p_\tau(j) + p_\tau(j+1) + p_\tau(j+2) + \ldots \qquad (j=0, 1, 2, \ldots k)$$

and

$$\ell_\tau(j) = p_\tau(-j) + p_\tau(-j-1) + p_t(-j-2) + \ldots \qquad (j=1, 1, 2, \ldots, k+1).$$

The matrix P^*_τ $(k+1)$ can be partitioned as

$$P^*_\tau(k+1) = \begin{bmatrix} 1 & L_\tau^T(k+1) \\ \underline{O}_\tau(k+1) & P_\tau(k+1) \end{bmatrix}$$

where O_τ $(k+1)$ is a zero column vector of size $(k+1)$, L^T_τ $(k+1)$ is the transpose of column vector L_τ $(k+1)$ or

$$L^T_\tau(k+1) = [\ell_\tau(-1)\,\ell_\tau(-2)\ldots\ell_\tau(-k)\,\ell_\tau(-k-1)]$$

and P_τ $(k+1)$ is the so-called 'restricted' matrix for season τ.

From the seasonal restricted transition matrix P_τ $(k+1)$ a product matrix $R^{(m)}_{k+1}$ is obtained as

$$R^{(m)}_{k+1} = \prod_{i=1}^{m} P_{\tau(i)}\,(k+1) \tag{8}$$

where m is the total number of seasons in the planning horizon, say, of n years, the subscript τ (i) is defined as

$$\tau(i) = (m+1-i) - [(m-i)/w]\;.\,w \tag{9}$$

where w the number of seasons in a year, and $[(m-i)/w]$ denotes the integer part of the number in the argument. The superscript in $R^{(m)}_{k+1}$ is used only to indicate the total number of matrices involved in the product.

The matrix $R^{(m)}_{(k+1)}$ with elements denoted by $r^{(m)}_{k+1}$ (i, j) gives the probabilities of transition from state j to state i in m steps. Hence, $\sum_{s=1}^{k+1} r^{(m)}_{k+1}$ $(s,\ k+1)$ is the probability that the reservoir initially full does not end up empty at the end of m seasons which is essentially equivalent to $P\,(D_m \leqslant k)$, i.e.

$$P\,(D_m \leqslant k) = \sum_{s=1}^{k+1} r^{(m)}_{k+1}\ (s,\ k+1) \tag{10}$$

From the above equation it follows that

$$P\,(D_m = k) = \sum_{s=1}^{k+1} r^{(m)}_{k+1}\ (s,\ k+1) - \sum_{s=1}^{k} r^{(m)}_{k}\ (s,\ k) \tag{11}$$

Equations (11) can be used for any non-stationary or seasonal net inputs provided that the seasonal inputs are independent.

Note that as the number of states of the discretized inputs increases, the size of the transition matrices as well as the amount of computations required also increases. Hence, a trade-off should be made between accuracy and costs.

3.2 First-Order Markov Seasonal Inputs

In determining the distribution of the maximum accumulated deficit D_m of seasonal correlated inputs, an attempt will be made to formulate the analysis in a manner similar to that used for the seasonal independent case described in the previous section.

Let the water stored in the reservoir s_t be augmented by the net input x_t in the time interval $(t, t+1)$. That is,

$$s_{t+1} = s_t + x_t \tag{12}$$

is the reservoir storage at the end of such time interval. Note that the pair of variables (s_t, x_t) are independent.

Recall from the previous section that when the inputs x_t are independent, the sequence of partial sums $[s_t]$ is a simple Markov chain. When the net inputs are serially correlated $[s_t]$ is no longer a simple Markov chain since the current net inputs depends on the previous inputs. The method used in this study is to build into the model the persistence pattern of the inflows, and at the same time maintain the Markovian structure of the system. The serially correlated inputs $[x_t]$ is represented by a Markov chain and the pair $[s_t, x_t]$ is treated as the state of the system which is defined by a bivariate Markov chain describing the simultaneous distribution of s_t and x_t at time t. In this approach, seasonality of inputs can readily be incorporated by using seasonal bivariate transition probability matrices.

Consider the case of w seasons per year. As in the case of seasonal independent inputs, the inflows $[x_t]$ into the reservoir and the storage level $[s_t]$ will each have a common distribution defined for each season. Let the reservoir be of size $k+1$, and let the state space of the reservoir be such that the empty state is an absorbing boundary and the full state is a reflecting boundary. For example, suppose that for each season the discrete net inputs $[x_t]$ can only assume values (−1, 0+1) with different marginal probabilities, for each season. Thus, $[s_t, x_t]$ follows a periodic bivariate Markov chain with transition probabilities defined by

$$
\begin{array}{r|l}
 & (0,-1)\ \ (0,0)\ \ (0,-1)\ldots\ (k+1,1)\ (k+1,0)\ \ (k+1,1) \\
\hline
(0,-1) & \\
(0,0) & \\
(0,1) & \\
(1,-1) & \\
p_\tau^*\ (k+1)(1,0) & \qquad\qquad p_\tau\ (si\,|\,uj) \qquad\qquad\qquad\qquad (13) \\
(1,1) & \\
\vdots & \\
(k+1,-1) & \\
(k+1,0) & \\
(k+1,1) &
\end{array}
$$

where $p_\tau\ (si\,/\,uj)\ =\ p_\tau\ (s_t{=}s,\ x_t{=}i\,/\,s_{t-1}{=}u,\ x_{t-1}{=}j)$

and $\tau = 1, 2, \ldots, w$.

The matrix $p_\tau^*\ (k+1)$ of eq. (13) can be partitioned as

$$
p_\tau^*\ (k+1) = \begin{bmatrix} A_\tau\ (0) & C_\tau\ (k+1) \\ B_\tau\ (k+1) & p_\tau\ (k+1) \end{bmatrix} \qquad (14)
$$

where A_τ (0) is a partitioned matrix of size $(a\text{x}a)$, a is the number of states of net inputs (a=3 in this case), corresponding to the transition of storage from empty to empty, B_τ $(k+1)$ is a matrix of size $[a(k+1)\text{x}a]$ corresponding to the transition from empty to non-empty, C_τ $(k+1)$ is a matrix of size $[a\text{x}a(k+1)]$ corresponding to the transition fron non-empty to empty, and p_τ $(k+1)$ *is the 'restricted'* bivariate transition matrix for season τ.

The distribution of the maximum deficit D_m for m seasons in the planning horizon can be determined from the product of m 'restricted' bivariate transition matrices. For any m, the element $r_{k+1}^{(m)}(si|k+1, j)$ is defined from the product matrix $R_{k+1}^{(m)}$ as

$$R_{k+1}^{(m)} = \prod_{i=1}^{m} p_{\tau(i)}(k+1) \tag{15}$$

where $\tau(i)$ is defined as in eq. (9). The (si, uj) entry in the matrix $R_{k+1}^{(m)}$ represents the transition $(s_m = s,\ x_m = i / s_0 = u,\ x_0 = j)$. Hence $\sum_{s=1}^{k+1} \sum_i \sum_j r_{k+1}^{(m)}(si|k+1, j)\ p(x_0 = j)$ is the probability that the reservoir initially full at the beginning of season 1 does not become empty in m seasons.

Thus, the distribution of D_m of seasonal correlated inputs is given by

$$p(D_m \leqslant k) = \sum_{s=1}^{k+1} \sum_i \sum_j r_{k+1}^{(m)}(si|k+1, j)\ p(x_0 = j) \tag{16}$$

from which it follows that

$$p(D_m = k) = \sum_{s=1}^{k+1} \sum_i \sum_j r_{k+1}^{(m)}(si\,|\,k+1, j)\, p(x_0 = j)$$

$$- \sum_{s=1}^{k+1} \sum_i \sum_j r_k^{(m)}(si\,|\,k, j)\, p(x_0 = j). \tag{17}$$

It is interesting to note the similarity in the form of equations (11) and (17). The basic difference is on how $r_{k+1}^{(m)}(s, k+1)$, for seasonal independent inputs, and $r_{k+1}^{(m)}(si|k+1, j)$, for seasonal correlated inputs, are defined as well as the inclusion of the probability of the initial state of the net input in the latter.

4. Analytical and Simulation Results

In actual situation the inflow into the reservoir is a continuous random variable which can assume any value in a finite range. For the practical use of the theory, however, the continuous variable is represented by a discrete variable. As the number of discrete states increases the accuracy of the discrete approximation of the continuous process also increases. But coupled with the increase in accuracy is the attendant increase in the amount of computations involved. Therefore, some trade-off must be made which will depend on the extent of the data and facilities as well as budget available.

To illustrate the method described earlier a 3-state discrete representation of the inflows to the reservoir and a 4-season year time frame are adopted herein. However, for actual technological application in an engineering scale, the method would apply to the more detailed and complex cases as well.

The maximum accumulated deficit is a random variable derived from the cumulative or partial sums of the seasonal net input variables. Since these variables have seasonal or periodic parameters, properties of the maximum deficit D, such as the expected value $E(D_m)$ and variance Var (D_m) for given sample size m (number of seasons), are functions of such parameters. In the absence of analytical expressions for $E(D_m)$ and Var (D_m), a numerical study of the distributional properties of D_m can be made. Results of the numerical study of $E(D_m)$ and Var (D_m) computed based on the probability distribution of D_m are also checked by data generation method.

4.1 Independent Seasonal Inputs

The numerical comparisons of the functional dependence of $E(D)$ and Var (D) of the 3-state, 4-season independent net inputs (-1, 0, +1) for different combinations of the seasonal parameters are shown in Figs. 1 and 2, respectively. The plot of $E(D_m)$ with m (number of seasons) in the case of constant variance show that the expected deficit is a smooth increasing function of m. In the case of periodic variance, $E(D_m)$ is an increasing periodic function of m with same period as that of the seasonal variance. Furthermore, as the periodicity of the seasonal variance increases, the difference of the periodic deficit also increases. It can also be observed that $E(D_m)$ of inputs with periodic (seasonal) variances oscillates around $E(D_m)$ of inputs with variance equal to the average of the periodic variances.

It appears that the periodicity in $E(D_m)$ tends to die out as m increases, with $E(D_m)$ *approaching the* mean value for the stationary case (case 1). Moreover, cases (3) and (4), which correspond to asymetric periodic variance, have the $E(D_m)$ to be either above or below the $E(D_m)$ of constant variance (case 1) depending on whether the periodic variance begins with a variance larger or smaller than the average variance, respectively. Of course, in all cases $E(D_m)$ will converge to $E(D_m)$ for constant variance as m increases. The same thing is true for Var (D_m) except that, as noted above, Var (D_m) for periodic variances are smaller than Var (D_m) for constant variance although convergence occurs as m increases.

The results obtained by the numerical method were also verified by generating simulated samples from which the mean and variance of deficits of seasonal inputs were computed. The computed values by simulation are plotted in the same figures corresponding to the different cases studied. The plots show the close agreement between the expected deficit and the variance of deficit of seasonal input series obtained by both the numerical and the data generation methods.

4.2 First-Order Markov Seasonal Inputs

The seasonal net input process with an underlying first-order autoregressive dependence structure is approximated by a Markov chain with seasonal or periodic transition probability matrix. For illustrative purposes consider a 4-season year, and 3-state correlated net inputs (-1,0,+1) and four combinations of seasonal parameters as follows:

(1) $\bar{\sigma}_T = 0.50$, $s(\sigma_T) = 0.00$, $\bar{\rho}_T = 0.30$, and $s(\rho_T) = 0.00$

(2) $\bar{\sigma}_T = 0.50$, $s(\sigma_T) = 0.1581$, $\bar{\rho}_T = 0.30$ and $s(\rho_T) = 0.00$

(3) $\bar{\sigma}_T = 0.50$, $s(\sigma_T) = 0.00$, $\bar{\rho}_T = 0.30$, and $s(\rho_T) = 0.20$

(4) $\bar{\sigma}_T = 0.50$, $s(\sigma_T) = 0.1581$ $\bar{\rho}_T = 0.30$, and $s(\rho_T) = 0.20$

In all cases considered the seasonal net inputs have mean zero.

It is also interesting to study the distribution of the standardized deficit of seasonal net inputs, i.e. the distribution of $D_m^* = (D_m - E[D_m])/\sqrt{\text{Var}(D_m)}$. Figure 3 gives the standardized exact density function of D_m for m=60 of the seasonal symmetric discrete independent net inputs $(-1, 0, +1)$ with $\overline{\sigma_\tau^2} = 0.40$ and $s(\sigma_\tau^2) = 0.20$, and the standardized asymptotic density. The plot shows that the asymptotic result is remarkably a good approximation when corrected for the first and second moments. Thus, if the approximate distribution of the maximum deficit of seasonal inputs is desired, it suffices to correct the standardized asymptotic distribution of deficit for the mean and variance of deficit.

The $E(D_m)$ computed numerically are shown in Fig. 4 for the four cases listed above. One characteristic shown is that the expected deficit of variables with periodic standard deviation and periodic correlation is in general and increasing periodic function of m and is higher than those with a constant standard deviation for $m \geqslant 12$. The plot of $E(D_m)$ against m for the case of constant standard deviation but periodic ρ_τ shows that it has a small effect on the periodicity of $E(D_m)$ compared to the case of constant correlation. However, it appears that this effect is greater when the standard deviation is periodic.

Figure 5 shows the comparison of the variance of deficit for the cases considered. Periodicity in either the standard deviation or autocorrelation yields increasing periodic variance of deficit as m increases. Cases (1) and (2) show the effect of the periodic standard deviation σ_τ on Var (D_m). Since σ_τ starts with a value smaller than $\overline{\sigma}$ (case 1), then Var (D_m) for case (2) is smaller than Var (D_m) for case (1) for $m \neq w, 2w, 3w, \ldots$, where $w = 4$. Cases (1) and (3) show the effect of the periodic correlation ρ_τ. Initially, Var (D_m) for case (3) fluctuates (periodically) around Var (D_m) for case (1), however, after a few cycles (about 3 cycles in this example) it appears that Var (D_m) for case (3) drops below that of case (1) except for m equal to multiple of w for which the values are the same. Cases (1) and (4) show the effect of both periodic σ_τ and periodic ρ_τ. It is clear that after a few cycles (2 cycles in this case) the Var (D_m) for periodic σ_τ and ρ_τ is larger than Var (D_m) for constant $\bar{\sigma}_T=\sigma$ and $p_T=\bar{p}$, except for $m=jw+w/2$ where $j \geqslant 2$ and $w=4$. It must be noted, however, that in all cases analyzed, the values of Var (D_m) must converge to the same as m approaches infinity.

5. *Summary and Conclusions*

Previous studies elsewhere (Gomide, 1975) have shown that deficit analysis of annual stationary series follows directly from the theory of Markov chains when the state space of the process is such that one boundary is reflecting and the other is absorbing. In this paper, the same approach is extended to the case of periodic seasonal series. It has been shown that the distribution of maximum deficit of seasonal inputs can also be determined following the theory of Markov chains with seasonal transition probabilities.

For the independent input case, deficit analysis of seasonal inflows involves the use of periodic univariate one-step transition probability matrices to represent the possible transition of storage level for each season. The case of seasonal first-order Markov inputs is treated in such a way that the simultaneous seasonal distribution of storage level and net input is represented by a seasonal bivariate Markov chain.

Some examples of the numerical approach to the distribution of maximum deficit for simple cases of discrete seasonal net inputs assuming a four-season year are given. However, the same general algorithm applies equally well to any number of seasons in a year and any number of discretized states of net inputs. Numerical study of the properties of the maximum deficit such as the mean and the variance show that they are also functions of the seasonal parameters of the series such as the seasonal means, seasonal standard deviations and seasonal correlations. The expected value and variance of deficit of seasonal inputs with periodic parameters are also periodic, and converge to the corresponding value of the mean and variance of deficit for stationary case as the sample size increases.

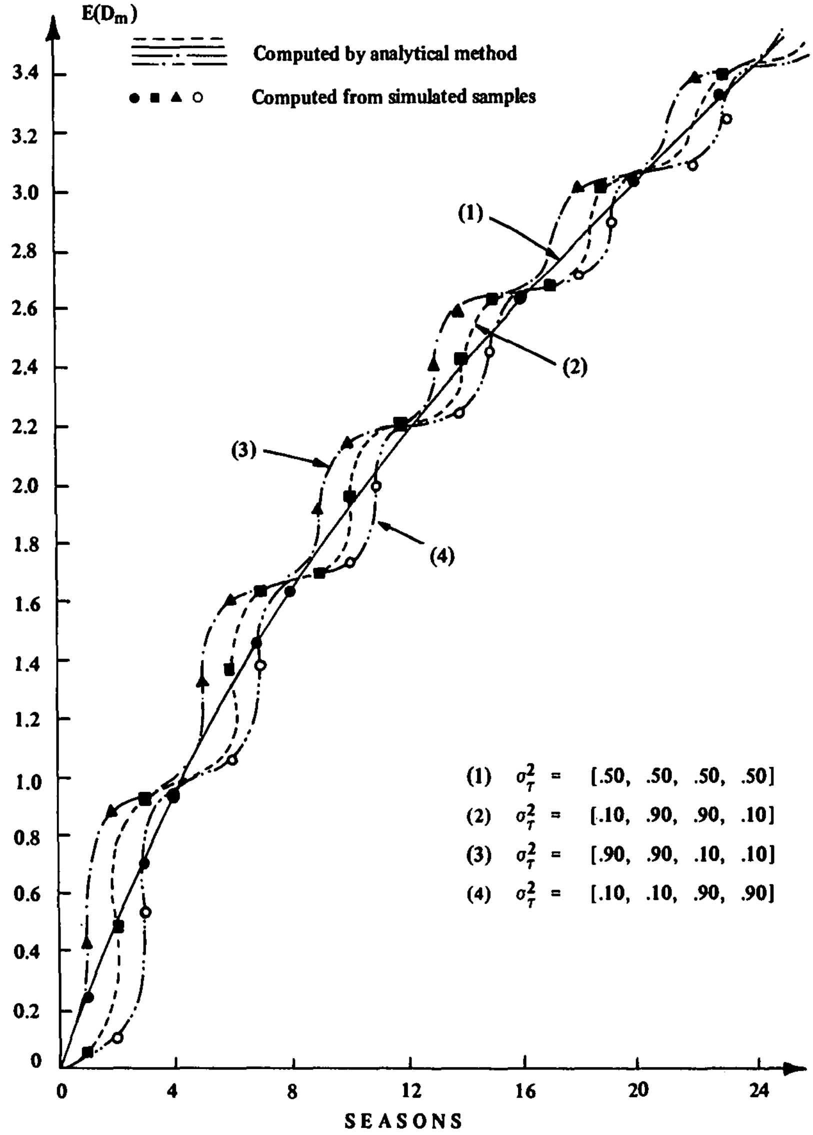

Figure 1. Expected deficit and the mean deficits obtained from simulated samples for the independent discrete seasonal net inputs (−1, 0, +1) with $\mu_T = 0$, $s(\mu_T) = 0.0$, and four sets of seasonal variances.

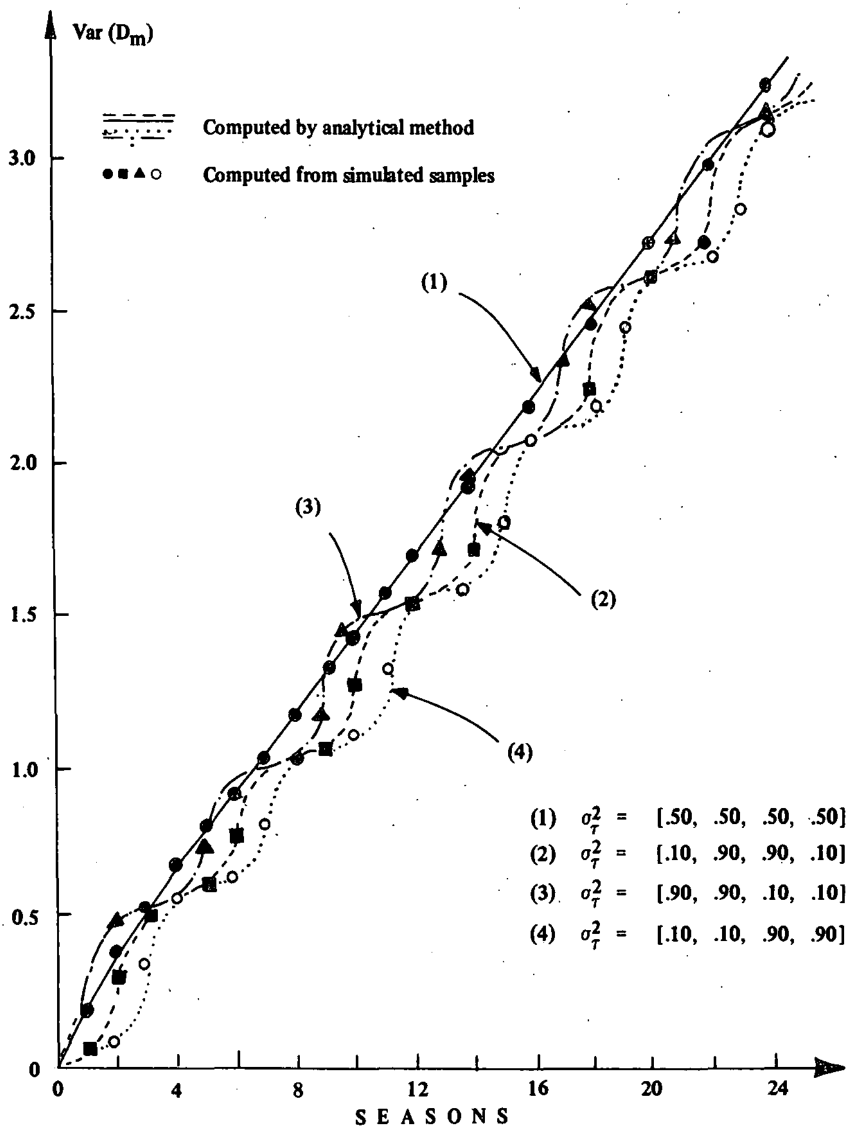

Figure 2. Variance of deficit obtained by the analytical method and computed from simulated samples for the independent discrete seasonal net inputs (−1, 0, +1) with μ_T = 0, s(μ_T) = 0.0, and four sets of seasonal variances.

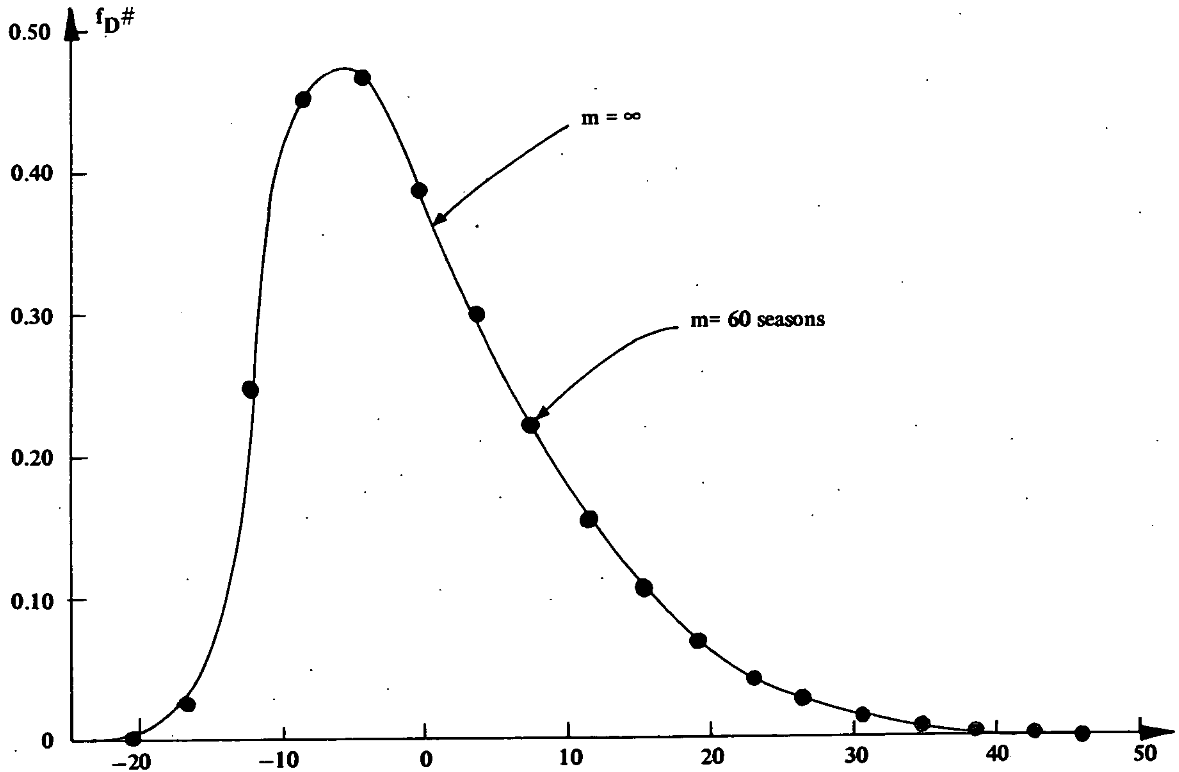

Figure 3. Distribution of $[D_m - E(D_m)] / \sqrt{Var(D_m)}$ for independent seasonal net inputs (−1, 0, +1) for m = 60, and m = ∞.

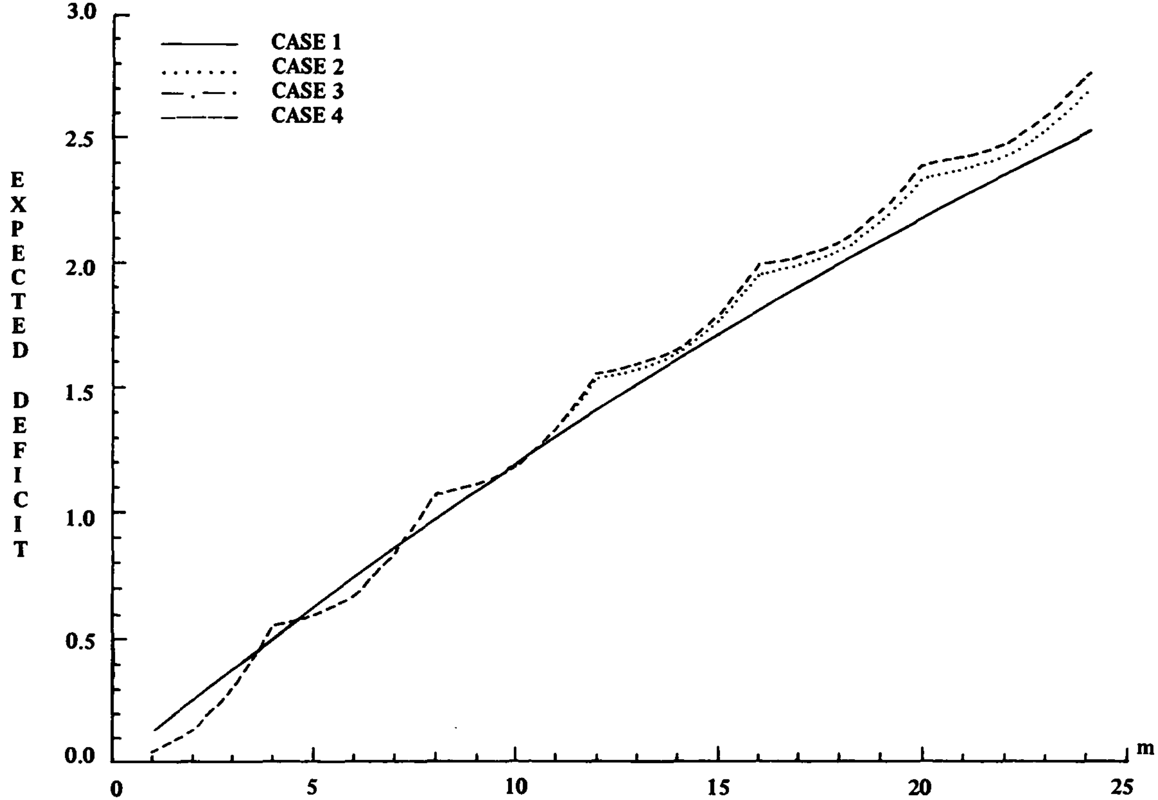

Figure 4. Expected deficit of the seasonal first-order Markov net inputs for four cases of parameter set.

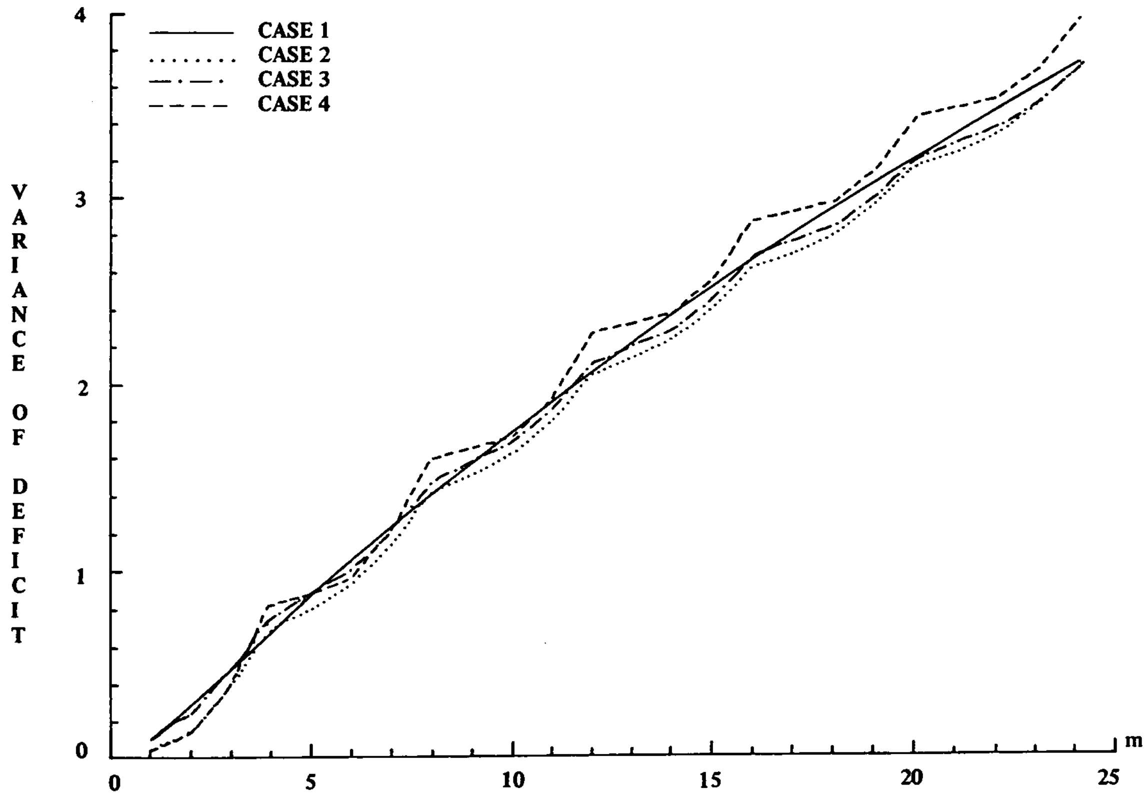

Figure 5. Variance of deficit of the seasonal first-order Markov net inputs for four cases of parameter set.

Storage Analysis of Seasonal Net Inputs

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